Differentiation

1. $y = ax^2 + bx + c$, $\frac{dy}{dx} = 2ax + b,\quad \left.\frac{dy}{dx}\right|_{x = x_0} = 2ax_0 + b$

$\therefore$ The tangent to the curve $y = ax^2 + bx + c$ at (x_0, y_0) is $y - y_0 = (2ax_0 + b)(x - x_0)$.

2. **(a)** $-\frac{2\cos x}{(\sin x - 1)^2}$ **(b)** $-\frac{2x}{\sqrt{1-x^2}\,(1+x^2)^{3/2}}$ **(c)** $\frac{1}{x}$ **(d)** $1 + \ln x$

(e) $x + 2x\ln x$ **(f)** $x^x(1 + \ln x)$ **(g)** $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ **(h)** $\frac{e^{ax}(1+ax)}{a}$

(i) $\frac{4e^{2x}}{(e^{2x}+1)^2}$ **(j)** $\frac{x+2a}{2x^{3/2}}\, e^{-\frac{a}{x}}$ **(k)** $\frac{k^{2x}+1}{k^{2x}-1}$ **(l)** $\frac{bnx^{n-1}}{(a+bx^n)\ln(a+bx^n)}$

(m) $-\frac{1}{x\sqrt{1-x^2}}$ **(n)** $\frac{1}{x\,\ln x\,\ln[\ln x]}$ **(o)** $\frac{x^3}{x^4 - 1 - \sqrt{1-x^4}}$

(p) $e^{(x^x)}(1+\ln x)$ **(q)** $e^{\sin x}\cos x$ **(r)** $e^x(1 - 2x^x - x^x\ln x)$

(s) $-\frac{a^{1+\frac{1}{x}}\ln a + ax^{\frac{1}{x}}\ln x - x^{1+\frac{1}{a}} - ax^{\frac{1}{x}}}{ax^2}$ **(t)** $\cot^2 x\csc x$

3. **(a)** $\frac{a+b-2abx}{(1-ax)^2(1-bx)^2}$ **(b)** $\frac{5x-7}{2\sqrt{(x-1)(x-2)}\,(x+1)^2}$

(c) $-\frac{bm+an+mx+nx}{(a+x)^{m+1}(b+x)^{n+1}}$ **(d)** $\frac{n}{x\sqrt{1-x^2}}\left(\frac{x}{1+\sqrt{1-x^2}}\right)^n$

4. $(1+x)^n = \sum_{k=0}^{n} C_k^n x^k$, differentiate we get: $n(1+x)^{n-1} = \sum_{k=1}^{n} kC_k^n x^{k-1}$, differentiate again, we have

$n(n-1)(1+x)^{n-2} = \sum_{k=2}^{n} k(k-1)C_k^n x^{k-2}$ and substitute $x = 1$, $\therefore$ $\sum_{k=2}^{n} k(k-1)C_k^n = n(n-1)2^{n-2}$.

5. $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} \Rightarrow y' = \lambda_1C_1e^{\lambda_1 x} + \lambda_2C_2e^{\lambda_2 x} \Rightarrow y'' = \lambda_1^2C_1e^{\lambda_1 x} + \lambda_2^2C_2e^{\lambda_2 x}$

$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = \lambda_1^2C_1e^{\lambda_1 x} + \lambda_2^2C_2e^{\lambda_2 x} - (\lambda_1+\lambda_2)(\lambda_1C_1e^{\lambda_1 x} + \lambda_2C_2e^{\lambda_2 x}) + \lambda_1\lambda_2(C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}) = 0.$

6. $y = x^n[C_1\cos(\ln x) + C_2\sin(\ln x)]$ …..(1)

Differentiate (1), $y' = -x^{n-1}[C_1\sin(\ln x) - C_2\cos(\ln x)] + n\,x^{n-1}[C_1\cos(\ln x) + C_2\sin(\ln x)]$

$\therefore$ $xy' = -x^n[C_1\sin(\ln x) - C_2\cos(\ln x)] + n\,y$ …..(2)

Differentiate (2), $xy'' + y' = -x^{n-1}[C_1\cos(\ln x) + C_2\sin(\ln x)] - nx^{n-1}[C_1\sin(\ln x) - C_2\cos(\ln x)] + ny'$,

Multiply by x, $x^2y'' + xy' = -y - n[xy' - ny] + nxy'$, by (1) and (2).

$x^2y'' + (1-2n)xy' + (1+n^2)y = 0.$

7. $y = 1 - \ln(x+y) + e^y$ $\frac{dy}{dx} = -\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) + e^y\frac{dy}{dx}$

$\therefore$ $\left(1 + \frac{1}{x+y} + e^y\right)\frac{dy}{dx} = -\frac{1}{x+y}$ $\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x+y+(x+y)e^y}$

8. **(a)** e^x **(b)** $ab^n e^{bx+c}$ **(c)** $(\ln a)^n a^x$ **(d)** $(-1)^{n-1}(n-1)!\frac{1}{x^n}$

(e) $u = e^x$, $v = 1/x$, By Leibnitz Theorem,

$$(uv)^{(n)} = \sum_{k=0}^{n} C_k^n u^{(n-k)} v^{(k)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} e^x (-1)^k k! \frac{1}{x^{k+1}} = \frac{e^x}{x^n} \sum_{k=0}^{n-1} (-1)^k P_k^n x^{n-1-k}$$

(f) $y = 2^x - \ln x$, $y' = (\ln 2)2^x - 1/x$, $y'' = (\ln 2)^2 2^x + 1/x^2$ $\therefore$ $y^{(n)} = (\ln 2)^n 2^x + (-1)^n \frac{(n-1)!}{x^n}$

(g) $u = e^{ax}$, $v = P_n(x)$, By Leibnitz Theorem, $(uv)^{(n)} = \sum_{k=0}^{n} C_k^n u^{(n-k)} v^{(k)} = \sum_{k=0}^{n} C_k^n a^{n-k} e^{ax} (P_n(x))^{(k)}$

(h) $y = \frac{\ln x}{x}$, $y' = \frac{1-\ln x}{x^2}$, $y'' = \frac{-3+2\ln x}{x^4}$, $y^{(3)} = \frac{11-6\ln x}{x^4}$, $y^{(4)} = \frac{-50+24\ln x}{x^5}$,

$y^{(5)} = \frac{274-120\ln x}{x^6}$, $y^{(6)} = \frac{-1764+720\ln x}{x^7}$ $\therefore$ $y^{(n)} = (-1)^{n+1} n! \frac{\sum_{i=1}^{n} \frac{1}{i} - \ln x}{x^{n+1}}$

(i) $y = \frac{1}{\sqrt{ax+b}}$, $y' = -\frac{a}{2(ax+b)^{3/2}}$, $y'' = \frac{3a^2}{4(ax+b)^{5/2}}$, $y^{(3)} = -\frac{15a^3}{8(ax+b)^{7/2}}$, $y^{(4)} = \frac{105a^4}{16(ax+b)^{9/2}}$

$y^{(n)} = (-1)^n \frac{(2n-1)!!a^n}{2n(ax+b)^{(2n+1)/2}}$, where $(2n-1)!! = (2n-1)(2n-3) \ldots (5)(3)(1)$.

(j) $y = x^4 \ln x$, $y' = x^3 + 4x^3 \ln x$, $y'' = 7x^2 + 12x^2 \ln x$, $y^{(3)} = 26x + 24x \ln x$, $y^{(4)} = 50 + 24 \ln x$
$y^{(5)} = 24x^{-1}$, $y^{(6)} = -24x^{-2}$, $y^{(7)} = 48x^{-3}$, $y^{(8)} = -144x^{-4}$,

$\therefore$ For $n \geq 5$, $y^{(n)} = (-1)^{n+1} \frac{24 \times (n-5)!}{x^{n-4}}$

9. $y = a_0 + a_1 x + \ldots + a_n x^n + e^x$. $\frac{d^n y}{dx^n} = a_n n! + e^x$.

10. $x - y + e^{x+y} = 2$, $1 - \frac{dy}{dx} + e^{x+y}\left[1 + \frac{dy}{dx}\right] = 0 \Rightarrow \frac{dy}{dx} = \frac{1+e^{x+y}}{1-e^{x+y}}$.

11. **(i)** $y = \sin^{-1} x \Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$ $\therefore \frac{d}{dx}\left(\sin^{-1} \frac{x^2}{(x^4+a^4)^{1/2}}\right) = \frac{2xa^2}{x^4+a^4}$

(ii) $\frac{x^2-2}{x^2-4}$

12. $x = \sin t$, $y = \sin pt$, $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = p\cos pt$, $\frac{dy}{dx} = \frac{p\cos pt}{\cos t}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\frac{p\cos pt}{\cos t}\frac{dt}{dx} = \frac{-p^2\cos t \sin pt + p\sin t \cos pt}{\cos^2 t} \Big/ \cos t = \frac{-p^2\cos t \sin pt + p\sin t \cos pt}{\cos^3 t}$$

$$\therefore \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = (1-\sin^2 t)\left(\frac{-p^2\cos t\sin pt + p\sin t\cos pt}{\cos^3 t}\right) - \sin t\left(\frac{p\cos pt}{\cos t}\right) + p^2(\sin pt) = 0$$

13. **(a)** $\frac{1}{x^2}\tan\frac{1}{x}$ **(b)** $2x\tan\left(\frac{\pi}{4}-x^2\right)$ **(c)** $\frac{1}{(1+x)\sqrt{1+2x}}$

(d) $2e^{\sin 2x}\cos 2x$ **(e)** $-\sin x\cos(\cos x)$ **(f)** $6xe^{3x^2}$

(g) $x^{\cos x}\left(\frac{\cos x}{x}-\log x\sin x\right)$ **(h)** $a^x \ln a$ **(i)** $\frac{2x}{\sqrt{1-x^4}\sin^{-1}x^2}$

(j) $\frac{1}{\sqrt{1-x^2}}$ **(k)** $2\sqrt{a^2-x^2}$ **(l)** $-\frac{2nx^{n-1}}{1+x^{2n}}$

(m) $\frac{\cos x}{2\sqrt{\sin x}\sqrt{1-\sin x}}$ **(n)** $2n\sin^n x\cos^n x\cot 2x$

14. **(a)** $x^2 + y^2 + 2gx + 2fy + c = 0 \quad \Rightarrow \quad 2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}\Big|_{(x_0,y_0)}=-\frac{x_0+g}{y_0+f}$

Tangent: $y-y_0=-\frac{x_0+g}{y_0+f}(x-x_0)$ or $xx_0+yy_0+g(x+x_0)+f(y+y_0)+c=0$

Normal: $y-y_0=\frac{y_0+f}{x_0+g}(x-x_0)$

(b) $y^2 = 4ax \quad \Rightarrow \quad 2y\frac{dy}{dx}=4a \quad \Rightarrow \frac{dy}{dx}\Big|_{(x_0,y_0)}=\frac{2a}{y_0}$

Tangent: $y-y_0=\frac{2a}{y_0}(x-x_0)$ or $yy_0=2a(x+x_0)$

Normal: $y-y_0=-\frac{y_0}{2a}(x-x_0)$

(c) $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \quad \Rightarrow \quad \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}\Big|_{(x_0,y_0)}=-\frac{b^2x_0}{a^2y_0}$

Tangent: $y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$ or $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$ Normal: $y-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$

(d) $\frac{x}{a}-\frac{y}{b}=1 \Rightarrow \frac{dy}{dx}\Big|_{(x_0,y_0)}=-\frac{b}{a}$ Tangent: $y-y_0=-\frac{b}{a}(x-x_0)$ or $\frac{x}{a}-\frac{y}{b}=1$

Normal: $y-y_0=\frac{a}{b}(x-x_0)$

(e) $xy = c^2 \quad \Rightarrow \quad \frac{dy}{dx}\Big|_{(x_0,y_0)}=-\frac{y_0}{x_0}$ Tangent: $y-y_0=-\frac{y_0}{x_0}(x-x_0)$ or $xy_0 + yx_0 = 2c^2$

Normal: $y-y_0=\frac{x_0}{y_0}(x-x_0)$

(f) $\sqrt{x}+\sqrt{y}=\sqrt{a} \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}\Big|_{(x_0,y_0)}=-\sqrt{\frac{y_0}{x_0}}$

Tangent: $y-y_0=-\sqrt{\frac{y_0}{x_0}}(x-x_0)$ Normal: $y-y_0=\sqrt{\frac{x_0}{y_0}}(x-x_0)$

15. Let P(n) : $\displaystyle\sum_{i=1}^{n}\sin ix = \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}$

P(1) is obviously true. Assume P(k) is true for some $k \in \mathbf{N}$, i.e. $\displaystyle\sum_{i=1}^{k}\sin ix = \frac{\sin\frac{(k+1)x}{2}\sin\frac{kx}{2}}{\sin\frac{x}{2}}$ …(*)

For P(k+1), $\displaystyle\sum_{i=1}^{k+1}\sin ix \underset{\text{by (*)}}{=} \frac{\sin\frac{(k+1)x}{2}\sin\frac{kx}{2}}{\sin\frac{x}{2}} + \sin(k+1)x = \frac{\sin\frac{(k+1)x}{2}\sin\frac{kx}{2}}{\sin\frac{x}{2}} + 2\sin\frac{(k+1)x}{2}\cos\frac{(k+1)x}{2}$

$$= \frac{\left[\sin\frac{kx}{2} + 2\cos\frac{(k+1)x}{2}\sin\frac{x}{2}\right]\sin\frac{(k+1)x}{2}}{\sin\frac{x}{2}} = \frac{\left[\sin\frac{kx}{2} + \left(\sin\frac{(k+2)x}{2} - \sin\frac{kx}{2}\right)\right]\sin\frac{(k+1)x}{2}}{\sin\frac{x}{2}}$$

$= \dfrac{\sin\frac{(k+2)x}{2}\sin\frac{(k+1)x}{2}}{\sin\frac{x}{2}}$ $\therefore$ P(k+1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ $n \in \mathbf{N}$.

Differentiate the above proposition, we have:

$$\sum_{k=1}^{n} k\cos kx = \frac{\sin\frac{x}{2}\left[\frac{n+1}{2}\cos\frac{(n+1)x}{2}\sin\frac{nx}{2} + \frac{n}{2}\cos\frac{nx}{2}\sin\frac{(n+1)x}{2}\right] - \frac{1}{2}\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}}$$

$$= \frac{\frac{n+1}{2}\sin\frac{x}{2}\left[\cos\frac{(n+1)x}{2}\sin\frac{nx}{2} + \cos\frac{nx}{2}\sin\frac{(n+1)x}{2}\right] - \frac{1}{2}\sin\frac{(n+1)x}{2}\left[\sin\frac{x}{2}\cos\frac{nx}{2} + \sin\frac{nx}{2}\cos\frac{x}{2}\right]}{\sin^2\frac{x}{2}}$$

$$= \frac{\frac{n+1}{2}\sin\frac{x}{2}\sin\frac{(2n+1)x}{2} - \frac{1}{2}\sin\frac{(n+1)x}{2}\sin\frac{(n+1)x}{2}}{\sin^2\frac{x}{2}} = \frac{\frac{n+1}{2}\sin\frac{x}{2}\sin\frac{(2n+1)x}{2} - \frac{1}{2}\sin^2\frac{(n+1)x}{2}}{\sin^2\frac{x}{2}}$$

16. $1 + x + x^2 + \ldots + x^n = \dfrac{x^{n+1}-1}{x-1} \underset{\text{Differentiate}}{\Rightarrow} 1 + 2x + 3x^2 + \ldots + nx^{n-1} = \dfrac{(x-1)(n+1)x^n - (x^{n+1}-1)}{(x-1)^2} = \dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$

$$\underset{\text{multiply x}}{\Rightarrow} x + 2x^2 + 3x^3 + \ldots + nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

$$\underset{\text{Differentiate}}{\Rightarrow} 1^2 + 2^2x + 3^2x^2 + \ldots + n^2x^n = \frac{n^2x^{n+2} - (2n^2+2n-1)x^{n+1} + (n+1)^2x^n - x - 1}{(x-1)^3}$$

17. $2^n \sin\frac{x}{2^n}\cos\frac{x}{2}\cos\frac{x}{4}....\cos\frac{x}{2^n} = 2^{n-1}\cos\frac{x}{2}\cos\frac{x}{4}....\cos\frac{x}{2^{n-1}}\left(2\sin\frac{x}{2^n}\cos\frac{x}{2^n}\right)$

$= 2^{n-1}\cos\frac{x}{2}\cos\frac{x}{4}....\cos\frac{x}{2^{n-1}}\sin\frac{x}{2^{n-1}} = 2^{n-2}\cos\frac{x}{2}\cos\frac{x}{4}...\cos\frac{x}{2^{n-2}}\left(2\cos\frac{x}{2^{n-1}}\sin\frac{x}{2^{n-2}}\right) = ... = \sin x$

$\therefore \quad \cos\frac{x}{2}\cos\frac{x}{4}....\cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\frac{x}{2^n}}$ (better proof can be carried out by induction rather than deduction)

Taking logarithm of the above, $\ln\cos\frac{x}{2} + \ln\cos\frac{x}{4} + ... + \ln\cos\frac{x}{2^n} = \ln\sin x - \ln\sin\frac{x}{2^n} - \ln 2^n$

Differentiate we get: $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + ... + \frac{1}{2^n}\tan\frac{x}{2^n} = \cot x - \frac{1}{2^n}\cot\frac{x}{2^n}$

18. $y = \frac{(n+1+x)^{n+1}}{(n+x)^n} \Rightarrow \ln y = (n+1)\ln(n+1+x) - n\ln(n+x) \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{n+1}{n+1+x} - \frac{n}{n+x} = \frac{x}{(n+1+x)(n+x)}$

$\therefore$ Since $x > 0$, $y > 0$, $\frac{dy}{dx} > 0$ and y is increasing with x.

$\therefore \quad \frac{(n+1+x)^{n+1}}{(n+x)^n} > \frac{(n+1+0)^{n+1}}{(n+0)^n} \qquad \therefore \quad \left(1+\frac{x}{n}\right)^n < \left(1+\frac{x}{n+1}\right)^{n+1}$

19. $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right) \quad \frac{dx}{dy}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(1\Big/\frac{dx}{dy}\right) = \frac{d}{dy}\left(1\Big/\frac{dx}{dy}\right)\frac{dy}{dx} = -\frac{\frac{d}{dy}\left(\frac{dx}{dy}\right)}{\left(\frac{dx}{dy}\right)^2}\frac{1}{\frac{dx}{dy}} = -\frac{\frac{d}{dy}\left(\frac{dx}{dy}\right)}{\left(\frac{dx}{dy}\right)^3}$$

20. $\frac{dy}{dx} = \frac{dy}{dt}\frac{dx}{dt} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dg(t)}{dt}\Big/\frac{df(t)}{dt}$

(a) P: $x = a(2\cos t + \cos 2t)$, $y = a(2\sin t - \sin 2t)$ $\quad\therefore \frac{dx}{dt} = -2a(\sin t + \sin 2t)$, $\frac{dy}{dt} = 2a(\cos t - \cos 2t)$

$\therefore$ Grad. of tangent $= \frac{dy}{dx} = -\frac{\cos t - \cos 2t}{\sin t + \sin 2t} = -\tan\frac{t}{2}$ …..(1)

Tangent at P: $y - a(2\sin t - \sin 2t) = -\tan\frac{t}{2}[x - a(2\cos t + \cos 2t)]$ …..(2)

Normal at P: $y - a(2\sin t - \sin 2t) = \cot\frac{t}{2}[x - a(2\cos t + \cos 2t)]$ …..(3)

(b) Q : $x_1 = a\left(2\cos\left(-\frac{t}{2}\right) + \cos 2\left(-\frac{t}{2}\right)\right) = a\left(\cos t + 2\cos\left(\frac{t}{2}\right)\right)$, $y_1 = a\left(\sin t - 2\sin\frac{t}{2}\right)$

R : $x_2 = a\left(2\cos\left(\pi - \frac{t}{2}\right) + \cos 2\left(\pi - \frac{t}{2}\right)\right) = a\left(\cos t - 2\cos\left(\frac{t}{2}\right)\right)$, $y_2 = a\left(\sin t + 2\sin\frac{t}{2}\right)$

$$m_{QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a\sin(t/2)}{-4a\cos(t/2)} = -\tan\frac{t}{2} \qquad \text{.....(4)}$$

$$m_{PQ} = \frac{y - y_1}{x - x_1} = \frac{a[\sin 2t - \sin t - 2\sin(t/2)]}{a[2\cos(t/2) - \cos t - \cos 2t]} = \frac{2\cos(3t/2)\sin(t/2) - 2\sin(t/2)}{2\cos(t/2) - 2\cos(3t/2)\cos(t/2)} = -\frac{\sin(t/2)}{\cos(t/2)} = -\tan\frac{t}{2} \qquad \text{.....(5)}$$

From (1), (4) and (5), we have: Grad. of tangent at P = $m_{PQ} = m_{QR}$ and P, Q, R are collinear.

the tangent at P meets the curve in the points Q, R whose parameters are $-\frac{1}{2}t$ and $\pi - \frac{1}{2}t$.

(c) $QR^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$$= [-4a\cos(t/2)]^2 + [4a\sin(t/2)]^2 = 16a^2[\cos^2(t/2) + \sin^2(t/2)] = 16a^2$$

$\therefore$ QR = 4a

(d) Replace t by (-t/2) and (π - t/2) in (2), we get:

Tangent at Q: $$y - a\left(-2\sin\frac{t}{2} + \sin t\right) = \tan\frac{t}{4}\left[x - a\left(2\cos\frac{t}{2} + \cos t\right)\right] \qquad \text{.....(6)}$$

Tangent at R : $$y - a\left(2\sin\frac{t}{2} + \sin t\right) = -\cot\frac{t}{4}\left[x - a\left(-2\cos\frac{t}{2} + \cos t\right)\right] \qquad \text{.....(7)}$$

Grad. of tangent at Q × Grad. of tangent at R = tan (t/4) × [- cot (t/4)] = -1

$\therefore$ the tangents are Q and R are at right angles.

By substituting in (6) and (7) respectively we can check that:

The point Y = $\left(a\cos\left(-\frac{t}{2}\right), a\sin\left(-\frac{t}{2}\right)\right) = \left(a\cos\frac{t}{2}, -a\sin\frac{t}{2}\right)$ is on tangent at Q.

and the point Z = $\left(a\cos\left(\pi - \frac{t}{2}\right), a\sin\left(\pi - \frac{t}{2}\right)\right) = \left(-a\cos\frac{t}{2}, a\sin\frac{t}{2}\right)$ is on tangent at R.

Let the tangent at Q and the tangent at R intersect at the point X.

$\therefore$ $\angle$YXZ = rt $\angle$. Also the mid-point of YZ , O = (0, 0), which is a constant point.

By the converse of $\angle$ in semicircle, the locus of point X is a circle centre O.

The radius of the circle = OY = $\sqrt{\left(a\cos\frac{t}{2}\right)^2 + \left(-a\sin\frac{t}{2}\right)^2} = a$.

$\therefore$ The tangents are Q and R intersect on the circle $x^2 + y^2 = a^2$.

(e) Replace t by (-t/2) and (π - t/2) in (3), we get:

Normal at Q: $$y - a\left(-2\sin\frac{t}{2} + \sin t\right) = -\cot\frac{t}{4}\left[x - a\left(2\cos\frac{t}{2} + \cos t\right)\right] \qquad \text{.....(8)}$$

Normal at R: $$y - a\left(2\sin\frac{t}{2} + \sin t\right) = \tan\frac{t}{4}\left[x - a\left(-2\cos\frac{t}{2} + \cos t\right)\right] \qquad \text{.....(9)}$$

The normals at P, Q, R are concurrent and meet at the point x = 3a cos t , y = 3a sin t.

This can be checked by either by substitution of the point in (3),(8) and (9) or solve any two of the equations (3), (8) and (9) to get the point. Obviously, $x^2 + y^2 = (3a\cos t)^2 + (3a\sin t)^2 = 9a^2$ and the normals at P, Q, R are concurrent and intersect on the circle: $x^2 + y^2 = 9a^2$.

(f) $(x^2 + y^2 + 12ax + 9a^2)^2$

$= [(a(2\cos t + \cos 2t))^2 + (a(2\sin t - \sin 2t))^2 + 12a\,(a(2\cos t + \cos 2t)) + 9a^2]^2$

$= a^4\,[4\cos^2 t + 4\cos t\cos 2t + \cos^2 2t + 4\sin^2 t - 4\sin t\sin 2t + \sin^2 2t + 24\cos t + 12\cos 2t + 9]^2$

$= a^4\,[5 + 4\,(\cos t\cos 2t \quad - \sin t\sin 2t) + 24\cos t + 12\cos 2t + 9]^2$

$= a^4\,[5 + 4\cos 3t + 24\cos t + 12\cos 2t + 9]^2$

$= a^4\,[5 + 4\,(4\cos^3 t - 3\cos t) + 24\cos t + 12\,(2\cos^2 t - 1) + 9]^2$

$= a^4\,[16\cos^3 t + 24\cos^2 t + 12\cos t + 2]^2$

$= 4a^4\,[8\cos^3 t + 12\cos^2 t + 6\cos t + 1]^2$

$= 4a^4[2\cos t + 1]^6$

$4a(2x + 3a)^3 = 4a\,[2(a(2\cos t + \cos 2t)) + 3a\,]^3$

$= 4a^4\,[4\cos t + 2\cos 2t + 3]^3$

$= 4a^4\,[4\cos t + 2\,(2\cos^2 t - 1) + 3]^3$

$= 4a^4\,[4\cos^2 t + 4\cos t + 1]^3$

$=4a^4[2\cos t + 1]^6$

$\therefore \quad (x^2 + y^2 + 12ax + 9a^2)^2 = 4a(2x + 3a)^3.$

For reference, the graph of the function is given below:

The value of a is taken to be 1.

The tangent problem at part (e) is also illustrated.